

Effect of an External Field on a Modification of Luttinger's Many-Fermion Model

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A modification is given of Luttinger's one-dimensional model of interacting fermions to incorporate both Fermi surfaces using one physical type of particle. The system is solved in the presence of an arbitrary external field, with linear response and pair correlation functions readily computed. These constitute an explicit realization of a relation associated with the random phase approximation.

KEY WORDS: Exact solution; one-dimensional; fermions; linear response; pair correlation.

1. INTRODUCTION

Analytic solutions of interacting many-fermion systems are notoriously difficult to obtain. One must normally settle for a caricature of the system of interest, in which care has been taken to retain enough to answer questions about some specific points, and these alone. One extreme class of systems in which progress can be made in this fashion is characterized by particle interactions due to collective modes that group themselves into small Lie algebras (see, e.g., Ref. 1). For one-dimensional systems, one can go much further and, e.g., encompass hard-core or even δ -function interactions.⁽²⁾ The fact that arbitrary interaction potentials can also be handled in the one-dimensional case was first observed by Tomonaga.⁽³⁾ This was in the context of a possibly realistic situation, that in which changes in fermion occupation are restricted to a small distance from the Fermi surface, so that only first-order changes in kinetic energy had to be considered.

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Luttinger⁽⁴⁾ then formalized this model and introduced two electron species (of opposite spin, but any dichotomic variable would do) whose changes would be localized about the two Fermi levels of a one-dimensional system.

When Luttinger solved his model with two kinds of particles, each particle could interact only with particles of the other type, but not with particles of the same type. Later, when Mattis and Lieb⁽⁵⁾ solved the model correctly, they were able to diagonalize the Hamiltonian even when they let all particles interact with each other. They called this choice “natural in field theory.”⁽⁶⁾ Implicit was the fact that two particle species were necessary to obtain excitations near the two ends of the Fermi sea, $\pm k_F$, where k_F is the Fermi momentum (we will conventionally choose $\hbar = 1$). Nobody, to our knowledge, has tried to reduce these two kinds of fermion to one, a task that is imperative prior to a generalization to three dimensions,^{(7),2} since there the sign in front of k_F is meaningless and one can go continuously from any point of the Fermi surface to any other. Our first objective in the present work is to show how this can be accomplished by mixing the two Fermi seas in their overlapping region. The cost will be a further modification of the kinetic energy.

What sort of information can we expect to obtain reliably from such models? Since we have not simply approximated the system Hamiltonian, it is not clear that any absolute quantitative results are meaningful, although qualitative ones—the maintenance of the Fermi surface under interaction, for example—may well be. Rather, one might expect relationships between observable quantities to be tolerably well represented, even though individual quantities may not be. In particular, our second objective is to solve completely the reaction of a Fermi fluid to an external field, the resulting inhomogeneity manifesting itself in a number of quantities. We seek relationships between such quantities that do not depend upon details of the particle kinetic energy, and perhaps not of the interaction either. This will allow us the option of introducing artificiality into either, and some consequences of exercising this option will be examined.

It is worth pointing out the peculiar role of second quantization—and the underlying infinite Fermi sea—in these models. The corresponding first-quantized models are notably devoid of structure (see Mattis and Sutherland⁽⁹⁾ as well as Orfanopoulos⁽⁸⁾) and the problem of relating these to realistic systems has proved dauntingly refractory.

² In the distorted three-dimensional version, the kinetic energy is built up via eight particle labels occupying eight overlapping momentum regions, which are spliced together. For a general discussion see Ref. 8.

2. THE ODD-EVEN VERSION OF THE MODEL

The original Hamiltonian of Luttinger,⁽⁴⁾ as modified by Mattis and Lieb,⁽⁵⁾ took the form of a kinetic one-body part

$$H_0 = c \sum_k (a_{1k}^* a_{1k} - a_{2k}^* a_{2k}) k \quad (2.1)$$

and a potential energy two-body part

$$H_1 = \frac{1}{2L} \sum_p \tilde{v}(p) [\rho_1(-p) + \rho_2(-p)] [\rho_1(p) + \rho_2(p)] \quad (2.2)$$

$$\tilde{v}(p) = \int_0^L v(x) e^{ixp} dx, \quad p/(2\pi/L) \text{ integer}$$

(which includes interaction self-energy). Imagining H_1 to be turned on from an H_0 eigenstate in which type 1 particles fill a Fermi sea below k_F , and type 2 particles a Fermi sea above $-k_F$, and using the anticommutation relations

$$\begin{aligned} \{a_{jk}^*, a_{j'k'}^*\} &= \{a_{jk}, a_{j'k'}\} = 0 \\ \{a_{jk}^*, a_{j'k'}\} &= \delta_{jj'} \delta_{kk'} \end{aligned} \quad (2.3)$$

it was shown that the density components

$$\rho_i(p) = \sum_k a_{i,k+p}^* a_{ik}, \quad i = 1, 2 \quad (2.4)$$

satisfy the commutation relations

$$[\rho_j(-p), \rho_{j'}(p')] = \delta_{jj'} \delta_{pp'} (-1)^{j-1} pL/2\pi \quad (2.5)$$

where L is the spatial period of the implicit periodic boundary conditions. In deriving (2.5), it is important to take systematically the limit as the underlying momentum space becomes infinite in domain.

Restricting our attention to spinless particles, we would now like to have just one type of particle, perhaps with two labels, that satisfies the following requirements: (i) in the absence of interaction, all levels between $-k_F$ and k_F should be filled, (ii) interactions should provoke transitions across both k_F and $-k_F$, (iii) particles with the same or different labels cannot occupy the same level, but they should interact normally with each other. Here is how this can be done.

We imagine that the underlying system we are trying to model has an even number N of particles and we provisionally expand our spatial

domain to the line from $-L$ to L . On this domain, the wave numbers are integer multiples of

$$k_0 = \pi/L \quad (2.6)$$

The two basic Fermi surface levels must now differ by k_0 , i.e., they may be taken as $k_F = \frac{1}{2}Nk_0$ and $-(k_F - k_0)$. The type 1 particles in the interactionless state are now taken as filling every other level from k_F down: k_F , $k_F - 2k_0$, $k_F - 4k_0, \dots, -\infty$, whereas the type 2 particles fill alternate levels from $-(k_F - k_0)$ on up: $-(k_F - k_0)$, $-(k_F - 3k_0)$, $-(k_F - 5k_0), \dots, \infty$. With interaction, the remaining levels $k_F + 2k_0$, $k_F + 4k_0, \dots$ become available to type 1, while $-(k_F + k_0)$, $-(k_F + 3k_0), \dots$ can be partially occupied by type 2. Note that type 1 occupies only even levels and type 1 odd, or vice versa, depending upon whether $N/2$ is even or odd. For definiteness, we suppose the former to be true, which we now enforce by the convention that

$$\begin{aligned} a_{1k} &= 0 & \text{unless } k/k_0 &= 0 \pmod{2} \\ a_{2k} &= 0 & \text{unless } k/k_0 &= 1 \pmod{2} \end{aligned} \quad (2.7)$$

and similarly for a_{jk}^* . The commutation relations

$$\{a_{jk}, a_{j'k'}^*\} = \delta_{jj'} \delta_{kk'} \quad (2.8)$$

are of course now satisfied, with the $\delta_{jj'}$ factor redundant, since at common k , either a_{jk} or $a_{j'k}^*$ must vanish unless $j = j'$.

To construct our model using (2.6)–(2.8), we first move all particles in the region $(-L, 0)$ by L so that they fall in the region $(0, L)$. This will leave the interaction on the space $(-L, L)$ unchanged if it is of period L . In other words, we have

$$H_1 = \frac{1}{2L} \sum_{p/k_0 \text{ even}} \tilde{v}(p) \rho(-p) \rho(p) \quad (2.9)$$

where $\tilde{v}(p)$ is the Fourier transform on $(0, L)$. Since only even p are required,

$$\begin{aligned} \rho(p) &= \rho_1(p) + \rho_2(p) \\ &= \sum_{k/k_0 \text{ even}} a_{1k+p}^* a_{1k} + \sum_{k/k_0 \text{ odd}} a_{2k+p}^* a_{2k} \end{aligned} \quad (2.10)$$

If the Fermi sea for type 1 at $k = -\infty$ is filled, and that for type 2 at $k = \infty$, then these density components are readily found to satisfy

$$[\rho_1(-p), \rho_1(p)] = [\rho_2(p), \rho_2(-p)] = pL/2\pi \quad (2.11)$$

as the only nonvanishing commutators.

For the kinetic energy, we, however, choose the full

$$\begin{aligned}
 H_0 &= H_{01} - H_{02} \\
 &= c \sum_{k/k_0 \text{ even}} ka_{1k}^* a_{1k} - c \sum_{k/k_0 \text{ odd}} ka_{2k}^* a_{2k}
 \end{aligned}
 \tag{2.12}$$

associated with the state of type 1 occupation below k_F , type 2 above $-(k_F - k_0)$ in the absence of H_1 . Since it is readily seen that

$$[H_0, \rho_i(\pm p)] = \pm p(-1)^{i-1} c \rho_i(\pm p)
 \tag{2.13}$$

H_0 has precisely the same commutation relations with the $\rho_i(p)$ as

$$T_0 = c \frac{2\pi}{L} \sum_{\substack{p > 0 \\ p/k_0 \text{ even}}} [\rho_1(p) \rho_1(-p) + \rho_2(-p) \rho_2(p)]
 \tag{2.14}$$

We observe that the conditions preceding (2.6) are now satisfied, and that indeed the transitions in the vicinity of the Fermi surfaces are always by multiples of $2k_0 = 2\pi/L$. The interpretation of H_0 on the space $(0, L)$ is, however, not very intuitive.

The solution of $H = H_0 + H_1$ now follows as in Mattis and Lieb. We write

$$H = T_0 + H_1 + (H_0 - T_0)
 \tag{2.15}$$

where $H_0 - T_0$ commutes with all $\rho_i(p)$, and then carry out the unitary transformation generated by

$$S = i \frac{2\pi}{L} \sum_{\substack{p \neq 0 \\ p/k_0 \text{ even}}} \phi(p) / p \rho_1(p) \rho_2(-p)
 \tag{2.16}$$

Since

$$e^{iS} \rho_i(p) e^{-iS} = \rho_i(p) \cosh \phi(p) + \rho_{3-i}(p) \sinh \phi(p)
 \tag{2.17}$$

it is readily verified that if ϕ is chosen to satisfy

$$\tanh 2\phi(p) = -v(p) / [2c\pi + v(p)]
 \tag{2.18}$$

then

$$\begin{aligned}
 e^{iS} H e^{-iS} &= \frac{2\pi c}{L} \sum_{\substack{p > 0 \\ p/k_0 \text{ even}}} (e^{-2\phi(p)} - 1) [\rho_1(p) \rho_1(-p) + \rho_2(-p) \rho_2(p)] \\
 &\quad + H_0 + \sum_{\substack{p > 0 \\ p/k_0 \text{ even}}} cp \left[\left(1 + \frac{\tilde{v}(p)}{\pi c} \right)^{1/2} - 1 \right] + \frac{1}{2L} \tilde{v}(0) \rho(0)^2
 \end{aligned}
 \tag{2.19}$$

Observing that $\rho_1(-p)$ and $\rho_2(p)$ (for $p > 0$) function, to within a constant, as boson annihilators, and that H_0 commutes with the rest of the Hamiltonian of (2.19), it follows that if $\phi(p) > 0$, the interaction shift in ground-state energy is simply

$$\Delta E = \sum_{\substack{p > 0 \\ p/k_0 \text{ even}}} cp \left[\left(1 + \frac{\tilde{v}(p)}{\pi c} \right)^{1/2} - 1 \right] + \frac{1}{2L} \tilde{v}(0) \rho(0)^2 \quad (2.20)$$

$\rho(0)$ can be chosen arbitrarily, but is fixed in a grand ensemble, in which $H - \mu\rho(0)$ is minimized.

3. EXTERNAL FIELD

Our new linear energy model (LEM) seems perhaps distressingly devoid of structure when taken from the point of view of the ground-state energy shift (2.20). This situation changes when we apply an external field, periodic on $[0, L]$,

$$H_2 = \frac{1}{L} \sum_{p/k_0 \text{ even}} \tilde{u}(-p) \rho(p) \quad (3.1)$$

The solution for $H_u = H + H_2$ is not appreciably more difficult than that for H alone. We again apply the decoupling transformation generated by S of (2.16), obtaining instead

$$\begin{aligned} e^{iS} H_u e^{-iS} &= \frac{2\pi c}{L} \sum_{\substack{p > 0 \\ p/k_0 \text{ even}}} e^{-2\phi(p)} [\rho_1(p) \rho_1(-p) + \rho_2(-p) \rho_2(p)] \\ &+ \frac{1}{L} \sum_{\substack{p > 0 \\ p/k_0 \text{ even}}} e^{\phi(p)} \{ \tilde{u}(-p) [\rho_1(p) + \rho_2(p)] \\ &+ \tilde{u}(p) [\rho_1(-p) + \rho_2(-p)] \} + H_0 - T_0 + \Delta E \quad (3.2) \end{aligned}$$

Another unitary transformation W now suffices to eliminate the linear terms in (3.2), namely

$$\rho_i(p) \rightarrow \rho_i(p) - \frac{e^{3\phi(p)}}{2\pi c} \tilde{u}(p) \quad \text{for } p \neq 0 \quad (3.3)$$

obviously of the form $W_{\rho_i(p)} W^{-1}$. If $e^{iS_2} = W e^{iS}$, then

$$\begin{aligned} e^{iS_2} H_u e^{-iS_2} &= \frac{2\pi c}{L} \sum_{\substack{p > 0 \\ p/k_0 \text{ even}}} (e^{-2\phi(p)} - 1) [\rho_1(p) \rho_1(-p) + \rho_2(-p) \rho_2(p)] \\ &+ H_0 + \Delta E - \sum_{\substack{p > 0 \\ p/k_0 \text{ even}}} \frac{1}{2\pi c L} e^{4\phi(p)} \tilde{u}(p) \tilde{u}(-p) + \frac{1}{L} \tilde{u}(0) \rho(0) \quad (3.4) \end{aligned}$$

Now of course, in the ground state,

$$\Delta E_u = \Delta E - \sum_{\substack{p>0 \\ p/k_0 \text{ even}}} \frac{e^{4\phi(p)}}{2\pi cL} \tilde{u}(p) \tilde{u}(-p) + \frac{1}{L} \tilde{u}(0) \rho(0) \quad (3.5)$$

What are the obvious consequences of (3.5)? To start, we can obtain the nonuniform density expectation by means of functional (in x space) or partial (in k space) differentiation, i.e., since

$$\rho(p) = L \partial H_2 / \partial \tilde{u}(-p) = L \partial (H + H_2) / \partial \tilde{u}(-p) \quad (3.6)$$

except at $p=0$, where $\rho(p)$ is still arbitrary, we have

$$n(p) = \langle \rho(p) \rangle = - \frac{e^{4\phi(p)}}{2\pi c} \tilde{u}(p) \quad (3.7)$$

It then follows that the linear response function has the simple form

$$\frac{\partial n(p)}{\partial \tilde{u}(q)} = - \frac{e^{4\phi(p)}}{2\pi c} \delta_{p,q} \quad (3.8)$$

here independent of external field.

The pair correlation is another characteristic quantity; it may be defined by

$$F(p, q) = (1/N) \langle [\rho(p) - n(p)][\rho(q) - n(q)] \rangle \quad (3.9)$$

Now we need the full expression (3.4), together with the fully transformed

$$e^{iS_2} \rho_i(p) e^{-iS_2} = \rho_i(p) \cosh \phi(p) + \rho_{3-i}(p) \sinh \phi(p) - \frac{e^{4\phi(p)}}{2\pi c} \tilde{u}(p) \quad (3.10)$$

so that

$$e^{iS_2} [\rho_i(p) - n_i(p)] e^{-iS_2} = \rho_i(p) \cosh \phi(p) + \rho_{3-i}(p) \sinh \phi(p) \quad (3.11)$$

as well as

$$e^{iS_2} [\rho(p) - n(p)] e^{-iS_2} = [\rho_1(p) + \rho_2(p)] e^{\phi(p)} \quad (3.12)$$

There are four possibilities to consider. First, we need $p > 0, q > 0$:

$$\begin{aligned} & [\rho_1(p) + \rho_2(p)] e^{\phi(p)} [\rho_1(-q) + \rho_2(-q)] e^{\phi(q)} \\ &= \rho_1(p) \rho_1(-q) e^{\phi(p) + \phi(q)} + \left[\rho_2(-q) \rho_2(p) + \frac{Lp}{2\pi} \delta_{p,q} \right] e^{\phi(p) + \phi(q)} \\ &+ [\rho_1(p) \rho_2(-q) + \rho_2(p) \rho_1(-q)] e^{\phi(p) + \phi(q)} \end{aligned}$$

and so find at once

$$F(p, q) = \frac{L |p|}{2\pi N} e^{2\phi(p)} \delta_{p,q} \quad (3.13)$$

The other three possibilities are carried out similarly, and in fact (3.13), with the absolute value as inserted, is valid for all.

The model relation implied by (3.8) and (3.13) is obvious. It is that

$$\frac{\partial n(p)}{\partial \tilde{u}(q)} = -\frac{2\pi}{cpq} n^2 F(p, q)^2 \quad (3.14)$$

where $n = N/L$ is the mean density of the underlying system. But what is c ? According to (2.12), the type 1 kinetic energy, normally $p^2/2m$, has been represented by $cp + \text{const}$. In the vicinity of $k_F = \frac{1}{2}N\pi/L$, we have

$$p^2/2m = k_F^2/2m + (k_F/m)(p - k_F) + \dots$$

leading to the identification $c = k_F/m$ or

$$c = n\pi/2m \quad (3.15)$$

thereby converting (3.14) to

$$\frac{\partial n(p)}{\partial \tilde{u}(q)} = -\frac{4mn}{pq} F(p, q)^2 \quad (3.16)$$

This is precisely the relation corresponding to the random phase or collective coordinate approximation to fluid structure,⁽¹⁰⁾ which does not correspond to a well-defined Hamiltonian. We see, then, that despite the blatant artificiality of the LEM—the reader will surely be aware of the strange external field independence and consequent diagonality of the separate relations (3.8) and (3.13)—it does give a model justification for the simple (3.16). Perhaps it is the general structure of (2.9), (2.12)—the first-quantized kinetic energy does indeed generate a small Lie algebra—rather than their specific form that leads to (3.16), but this is far from obvious.

4. CONCLUSION

We have shown that it is possible to set up a version of the solvable one-dimensional fermion model of Tomonaga, Luttinger, Mattis, and Lieb, in which the reference system has a full Fermi surface, and in the context of only one physical type of particle. The cost is the requirement of a further modified kinetic energy, corresponding to the folding of the physical region

$[-L, L]$ onto $[0, L]$. However, it is still easy to relate this to the underlying Newtonian kinetic energy, and a previous approximate relationship between linear response and pair correlation now emerges as an exact result. Importantly, the technique can be generalized to three dimensions,^(7,8) where an extended odd-even version is absolutely crucial in avoiding a proliferation of fermion species. This will be reported in a future publication.

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